Large Petermann factor in chaotic cavities with many scattering channels

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Abstract. –
The quantum-limited linewidth of a laser cavity is enhanced above the Schawlow-Townes value by the Petermann factor K, due to the non-orthogonality of the cavity modes. The average Petermann factor $\langle K \rangle$ in an ensemble of cavities with chaotic scattering and broken time-reversal symmetry is calculated non-perturbatively using random-matrix theory and the supersymmetry technique, as a function of the decay rate Γ of the lasing mode and the number of scattering channels N. We find for $N \gg 1$ that for typical values of Γ the average Petermann factor $\langle K \rangle \propto \sqrt{N} \gg 1$ is parametrically larger than unity.

The study of resonant scattering goes back to the early work of Breit and Wigner [1], and was developed extensively in the context of nuclear physics [2]. The Breit-Wigner resonance is described by a frequency-dependent scattering matrix $S(\omega)$ with elements

$$S_{nm} = \delta_{nm} + \sigma_n \sigma'_m (\omega - \Omega + i\Gamma/2)^{-1} \ , \tag{1}$$

where Ω and Γ are, respectively, the center and the width of the resonance, and σ_n, σ'_m are the complex coupling constants of the resonance to the scattering channels n, m. (In the presence of time-reversal symmetry the scattering matrix is symmetric, hence $\sigma_n = \sigma'_n$.) The coupling constants of a Breit-Wigner resonance are related to its width by the sum rule [3]

$$\sum_{n,m} |\sigma_n \sigma'_m|^2 = \Gamma^2 \ . \tag{2}$$

The physics behind this sum rule is that narrow resonances are quasi-bound states requiring a weak coupling to the continuum.

The Breit-Wigner theory holds in the limit $\Gamma \to 0$ only. More generally, one may consider a pole $\Omega - i\Gamma/2$ somewhere in the lower half of the complex ω-plane and ask for the form of

the scattering matrix near that pole. The separable form (1) remains valid, even if Γ is not small. The sum rule (2), however, should be replaced by [4]

$$\sum_{n,m} |\sigma_n \sigma'_m|^2 = K\Gamma^2 , \quad K \geq 1 . \tag{3}$$

The total coupling strength of the resonance is larger than required by its width by a factor K. This factor has been studied theoretically and experimentally in laser cavities [5, 6, 7, 8], where it governs the quantum-limited linewidth,

$$\delta\omega = K\frac{\Gamma^2}{2I} , \tag{4}$$

of the radiation from the lasing mode (with output power I). The linewidth is larger than the Schawlow-Townes value $\Gamma^2/2I$ [9] by the factor K. In this context K is called the Petermann factor or excess noise factor [5].

The optical cavities considered in these studies typically have a simple shape, that is characterized by integrable rather than chaotic wave dynamics [10]. Recently, three of us investigated the statistics of the Petermann factor in the presence of chaotic scattering [4]. That work was restricted to the case of a single scattering channel ($N = 1$), meaning a scalar scattering matrix. If the cavity is coupled to the outside world via an opening of area $\mathcal{A}$, then a single scattering channel requires $\mathcal{A} \lesssim \lambda^2/2\pi$, with λ the wavelength of the radiation. The factor K is only slightly larger than 1 for such an almost closed cavity, and measuring the deviation from unity would be quite problematic. For a larger K one has to increase $\mathcal{A}$, thereby increasing the number of scattering channels $N \approx 2\pi\mathcal{A}/\lambda^2$.

The fundamental question addressed in this paper is how K increases with increasing N. We present for the case of broken time-reversal symmetry a non-perturbative calculation of the average Petermann factor $\langle K \rangle$ in an ensemble of chaotic cavities, valid for arbitrary N and Γ. A non-perturbative calculation is necessary because the mode selected for laser action has a decay rate Γ of the order of the typical smallest rate $\Gamma_0 = TN\Delta/2\pi$, with Δ the mean spacing in frequency of the resonances and T the transmission probability of the channels through the opening. (The time $1/\Gamma_0$ is the mean dwell time of photons in the cavity.) Existing methods of large-N perturbation theory [11] require that $\Gamma > \Gamma_0$, and are therefore not applicable to this problem. Our non-perturbative method is based on the supersymmetry technique [12, 13], applied to eigenvector correlations of non-Hermitian random matrices. Our conclusion is that $\langle K \rangle \propto \sqrt{N} \gg 1$ can become parametrically large for $N \gg 1$. The square-root increase is the outcome of a subtle balance between the smallness of Γ and the largeness of N, and could not have been anticipated from perturbative arguments, which would imply a linear increase.

To formulate the problem within the framework of random-matrix theory we proceed as in ref. [4]. The modes of a closed chaotic cavity, in the absence of time-reversal symmetry, are eigenvalues of an $M \times M$ Hermitian matrix H with independent Gaussian distributed elements [10, 14]. The limit $M \to \infty$ at fixed spacing Δ of the modes is taken at the end of the calculation. The opening in the cavity is described by an $M \times N$ coupling matrix W. Modes of the open cavity are complex eigenvalues of the non-Hermitian matrix $\mathcal{H} = H - i\pi WW^\dagger$ which determines the scattering matrix [2, 14]

$$S = 1 - 2\pi i W^\dagger (\omega - \mathcal{H})^{-1} W . \tag{5}$$

The matrix $\mathcal{H} = U \operatorname{diag}(z_1, z_2, \ldots, z_M)U^{-1}$ has complex eigenvalues z_j and the matrix U contains the right eigenvectors of $\mathcal{H}$ in its columns. (The matrix U^{-1} contains the left eigenvectors.) Because $\mathcal{H}$ is not Hermitian, U is not unitary and the left or right eigenvectors are not orthogonal among themselves. If the cavity is filled with a homogeneous amplifying

medium (amplification rate $1/\tau_a$) one has to add a term $i/2\tau_a$ to the eigenvalues, shifting them upwards towards the real axis, while the eigenvectors are unchanged. The lasing mode is the eigenvalue $z_l = \Omega - i\Gamma/2$ closest to the real axis, and the laser threshold is reached when the decay rate Γ of this mode equals the amplification rate $1/\tau_a$. Near the laser threshold it is only necessary to retain the contribution from the lasing mode. Combining eqs. (1), (3), and (5) one finds that the Petermann factor is given by [4, 6]

$$K = (U^\dagger U)_{ll} \, (U^{-1} U^{-1\dagger})_{ll} \; . \tag{6}$$

This formula shows that deviations of K from unity result from the non-unitarity of U, hence from the non-orthogonality of the cavity modes [6].

We will calculate the average Petermann factor at a given value of Γ and Ω. Following refs. [11, 15], we write this conditional average in the form

$$
\begin{aligned}
\langle K \rangle &= \Big\langle \sum_j (U^\dagger U)_{jj} \, (U^{-1} U^{-1\dagger})_{jj} \, \delta(z - z_j) \Big\rangle \Big\langle \sum_j \delta(z - z_j) \Big\rangle^{-1} \\
&= \frac{1}{\pi\rho(z)} \lim_{\varepsilon \to 0} \Big\langle \Big(\operatorname{tr} \frac{\varepsilon}{(z - \mathcal{H})(z^* - \mathcal{H}^\dagger) + \varepsilon^2} \Big)^2 \Big\rangle \; ,
\end{aligned}
\tag{7}
$$

where $z = \Omega - i\Gamma/2$. The eigenvalue density $\rho(z) = \langle \sum_j \delta(z - z_j) \rangle$ has been determined by Fyodorov and Sommers [13],

$$\rho(z) = (2\pi/\Delta^2)\mathcal{F}_1(\pi\Gamma/\Delta) \, \mathcal{F}_2(\pi\Gamma/\Delta) \; , \tag{8}$$

$$\mathcal{F}_1(y) = \frac{1}{(N-1)!} \, y^{N-1} \, e^{-gy} \quad , \quad \mathcal{F}_2(y) = (-1)^N e^{gy} \left(\frac{d}{dy} \right)^N \left(e^{-gy} \frac{\sinh y}{y} \right) \; . \tag{9}$$

The parameter $g \geq 1$ is related to the transmission probability T of a scattering channel by $g = 2/T - 1$.[1]

The evaluation of the quantity (7) proceeds by the supersymmetry approach. We introduce the generating functional $\langle \det(A - J) / \det(A + J) \rangle$, where J is a source matrix and

$$A \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} i\varepsilon & z - \mathcal{H} \\ z^* - \mathcal{H}^\dagger & i\varepsilon \end{pmatrix} \; . \tag{10}$$

The quantity $\pi\rho(z)\langle K \rangle = -\langle \operatorname{tr}[(A^{-1})_{11}] \, \operatorname{tr}[(A^{-1})_{22}] \rangle$ can be expressed in terms of derivatives of the generating functional with respect to J. Following the technique of supersymmetry [12, 13] one can express the Petermann factor as an integral over a 4×4 matrix Q,

$$\rho(z)\langle K \rangle = \frac{\pi}{4\Delta^2} \lim_{\varepsilon \to 0} \int dQ \, e^{-\mathcal{L}(Q)} \operatorname{Str}(Q\sigma_x P_0 P_+) \operatorname{Str}(Q\sigma_x P_0 P_-) \; , \tag{11}$$

$$\mathcal{L}(Q) = -\frac{\pi\varepsilon}{\Delta} \operatorname{Str}(Q\sigma_x) - \frac{\pi\Gamma}{2\Delta} \operatorname{Str}(\sigma_z Q) + N\operatorname{Str}\ln(1 + wQ\sigma_z) \; , \tag{12}$$

where w is related to T by $T = 4w(1 + w)^{-2}$. The matrix Q obeys the non-linear constraint $Q^2 = 1$ and belongs to the coset space of the unitary non-linear σ-model [12]. It is a "supermatrix", meaning that it has an equal number of commuting and anticommuting variables. The symbol "Str" denotes the graded trace of a supermatrix, $\operatorname{Str} A = \operatorname{tr} P_0 A$, where

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[1] We assume for simplicity that each channel has the same g. Expressions for $\mathcal{F}_{1,2}$ for channel-dependent g_n can be found in ref. [13], and our main result (14) given below remains valid for the general case.

$P_0 = \text{diag}(1, -1, 1, -1)$. The Pauli-matrices σ_x, σ_z are embedded in the 4×4 supermatrix space as $\sigma_z = \text{diag}(1, 1, -1, -1)$ and $(\sigma_x)_{jk} = \delta_{j,k+2} + \delta_{j+2,k}$. Furthermore, $P_\pm = \frac{1}{2}(1 \pm \sigma_z)$.

The integral (11) can be evaluated by introducing Efetov's parameterization of the matrix Q for the unitary σ-model [12]. The result is([2])

$$
\rho(z)\langle K \rangle = -\frac{\pi}{2\Delta^2} \lim_{\varepsilon \to 0} \varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_1^\infty d\lambda_1 \int_{-1}^1 d\lambda_2 \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \left(\frac{g + \lambda_2}{g + \lambda_1} \right)^N
$$
$$
\times J_0 \left(\varepsilon \sqrt{\lambda_1^2 - 1} \right) \exp\left[\frac{\pi\Gamma}{\Delta}(\lambda_1 - \lambda_2) \right] , \tag{13}
$$

where J_0 is a Bessel function. The remaining two ordinary integrals can be carried out as in the corresponding calculation of $\rho(z)$ [13]. We finally obtain

$$
\langle K \rangle = 1 + \frac{2\, S(y)}{\mathcal{F}_1(y)\mathcal{F}_2(y)} \quad , \quad S(y) = -\int_0^y dx\; \mathcal{F}_1(x) \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{F}_2(x) \quad , \quad y = \pi\Gamma/\Delta . \tag{14}
$$

A more explicit expression can be obtained by successive partial integrations:

$$
S(y) = \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} y^k \left(\frac{\mathrm{d}}{\mathrm{d}y} \right)^k \left(e^{-gy} \frac{\mathrm{d}}{\mathrm{d}y} \frac{\sinh y}{y} \right) . \tag{15}
$$

For $N = 1$ and $\Gamma \ll \Delta$ we recover the single-channel result of ref. [4], $\langle K \rangle = 1 + 2\pi\Gamma/3g\Delta$. Eq. (14) generalizes this result to any value of N and Γ.

To study the limit $N \gg 1$, it is convenient to derive integral expressions by replacing $\sinh y/y$ in $\mathcal{F}_2$ with $\frac{1}{2}\int_{-1}^1 d\lambda\, e^{-\lambda y}$. This was done in ref. [13] for the eigenvalue density, with the result

$$
\mathcal{F}_1(y)\mathcal{F}_2(y) = \frac{1}{2\, y^2\, (N-1)!} \int_{(g-1)y}^{(g+1)y} dt\; t^N\, e^{-t} . \tag{16}
$$

Similarly, we find for the function $S(y)$ that determines the Petermann factor the expression

$$
S(y) = -\frac{1}{4\, y^2\, (N-1)!} \int_{(g-1)y}^{(g+1)y} dt\; [t - (g-1)y][t - (g+1)y]\, t^{N-1}\, e^{-t} . \tag{17}
$$

For simplicity we restrict ourselves in what follows to the case $T = g = 1$ of fully transmitted scattering channels. When $N \gg 1$, the factor $t^N e^{-t}$ has a sharp maximum at $t = N$ and one can apply the saddle-point method to evaluate the t-integration. This gives the eigenvalue density [13] $\rho(z) \approx N/\pi\Gamma^2$ for $\Gamma > \Gamma_0$ (with $\Gamma_0 = N\Delta/2\pi$ for $T = 1$). The density vanishes exponentially as Γ drops below Γ_0. Application of the same saddle-point method to eq. (15) gives $\langle K \rangle \approx (2\pi/\Delta)(\Gamma - \Gamma_0)$. If $\Gamma - \Gamma_0 = \mathrm{O}(\Gamma_0)$, then this estimate would imply that $\langle K \rangle \propto N$. This linear scaling with N breaks down in the cutoff regime $\Gamma \lesssim \Gamma_0$, which is precisely the relevant regime for the laser cavities.

In order to study the cutoff regime we define the rescaled decay rate $u = \sqrt{N/2}(\Gamma/\Gamma_0 - 1)$ and take the limit $N \gg 1$ at fixed u. Expanding the integrands in eqs. (16) and (17) around the saddle-point and keeping the first non-Gaussian correction, we obtain

$$
\langle K \rangle = \sqrt{2N}[F(u) + u] + F(u) \left[2u + \tfrac{4}{3}u^3 + \tfrac{4}{3}(1 + u^2)\, F(u) \right] + \mathrm{O}(N^{-1/2}) , \tag{18}
$$

([2]) For Γ real positive the integrals (11) and (13) are formally divergent. The proper procedure for the further evaluation [13] is to choose Γ imaginary and to perform the analytic continuation at the end of the calculation.

where we have abbreviated $F(u) = \pi^{-1/2} \exp[-u^2][1 + \mathrm{erf}\,(u)]^{-1}$. The eigenvalue density in the cutoff regime has the form $\rho = (2\pi/N\Delta^2)[1 + \mathrm{erf}\,(u)] + \mathrm{O}(N^{-3/2})$. For $u = 0$ (hence $\Gamma = \Gamma_0$) eq. (18) simplifies to

$$\langle K \rangle_{u=0} = \sqrt{\frac{2N}{\pi}} + \frac{4}{3\pi} \; , \tag{19}$$

while for $u \ll -1$ (hence $\Gamma - \Gamma_0 \ll -\sqrt{N}\Delta$) we find

$$\langle K \rangle_{u \ll -1} = -\frac{1}{u}\sqrt{\frac{N}{2}} = \frac{\Gamma_0}{\Gamma_0 - \Gamma} \; . \tag{20}$$

To determine how $\langle K \rangle$ scales with N we need to estimate the most probable value of Γ for the lasing mode. The decay rate Γ_l of the lasing mode is the smallest among the cavity modes that are amplified by stimulated emission. The number m_a of amplified modes is $\gg N$ in the "good cavity" regime[3]. In a simplified model [16], we can determine Γ_l as the smallest of m_a independent random variables $\Gamma_1, \ldots, \Gamma_{m_a}$ with probability distribution $P(\Gamma) = (\Delta/2)\rho(z) \approx (1/2\Gamma_0)[1 + \mathrm{erf}\,(u)]$. The distribution of Γ_l is then given by

$$P_l(\Gamma_l) = m_a\, P(\Gamma_l) \left[1 - \int_0^{\Gamma_l} d\Gamma\, P(\Gamma)\right]^{m_a - 1} . \tag{21}$$

This distribution has a pronounced maximum at a value $u_{\max}$ determined by

$$\frac{\exp[-u_{\max}^2]}{[1 + \mathrm{erf}(u_{\max})]^2} = \frac{m_a - 1}{2}\sqrt{\frac{\pi}{2N}} \; . \tag{22}$$

If m_a is comparable to $\sqrt{N}$, we have $u_{\max} = \mathrm{O}(1)$ and the Petermann factor is to close to its value at $u = 0$, eq. (19). For $m_a \gg \sqrt{N}$ (which is realized in the good-cavity regime) we find $u_{\max} \sim -\sqrt{\ln(m_a/\sqrt{N})} \sim -\sqrt{\ln m_a} \ll -1$ and

$$\langle K \rangle \sim \sqrt{\frac{N}{\ln m_a}} \; . \tag{23}$$

This is still parametrically larger than unity as long as $m_a \ll e^N$.

We now compare our analytical findings with the results of numerical simulations. We generated a large number of random matrices $\mathcal{H}$ with dimension $M = 120$ ($M = 200$) for $N = 2, 4, 6, 8$ ($N = 10, 12$). Figure 1 shows the average of K at given Γ. We find excellent agreement with our analytical result (14). The scaling $\langle K \rangle \propto \sqrt{N}$ at $\Gamma = \Gamma_0$ is shown in fig. 2. The inset depicts the distribution of K at $\Gamma = \Gamma_0$ for $N = 10$, which can only be accessed numerically. We see that the mean Petermann factor is somewhat larger than the most probable value.

So far we only discussed the case of broken time-reversal symmetry, because this greatly simplifies the analysis. From a numerical simulation with real symmetric H (shown as well in fig. 2) we find that the average Petermann factor for $\Gamma = \Gamma_0$ is parametrically large also in the presence of time-reversal symmetry. The average increases again sublinearly with N, but with an exponent that is smaller than in the case of broken time-reversal symmetry.

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[3] A "good cavity" has a smallest typical decay rate (inverse mean dwell time) Γ_0 that is small compared to the amplification bandwidth Ω_a. Since $m_a \simeq \Omega_a/\Delta$ and $\Gamma_0 = N\Delta/2\pi$, it follows that $m_a \gg N$.
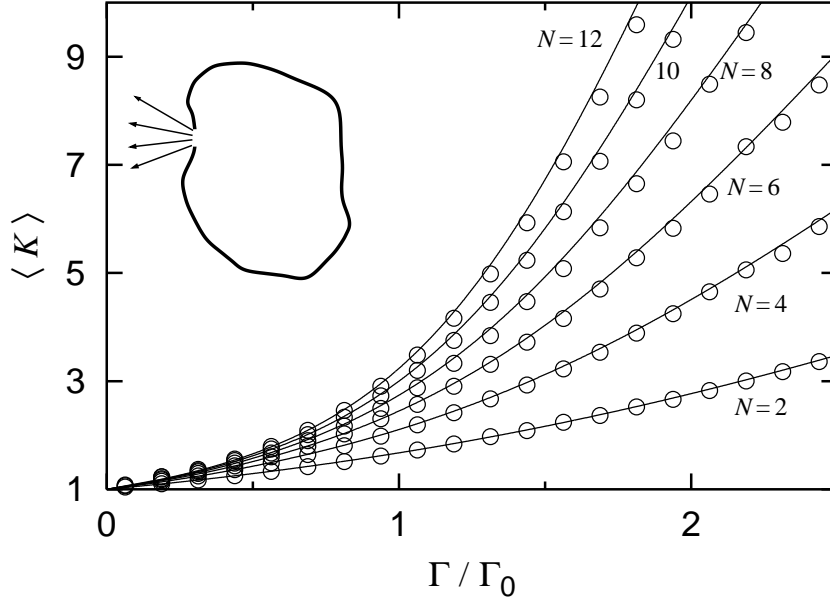
Fig. 1. – Average Petermann factor $\langle K \rangle$ as a function of the decay rate Γ (in units of $\Gamma_0 = N\Delta/2\pi$) for a chaotic cavity (inset) having an opening which supports N fully transmitted scattering channels. The solid curves are the analytical result (14), the data points are a numerical simulation.

In summary, we have calculated the average Petermann factor $\langle K \rangle$ for a chaotic cavity in the case of broken time-reversal symmetry. We find that for typical values of the decay rate Γ the average $\langle K \rangle \propto \sqrt{N}$ scales with the square root of the number of scattering channels N, hence with the square root of the area $\mathcal{A}$ of the opening of the cavity. This result could only be obtained by a non-perturbative calculation because laser action selects a mode with an untypically small decay rate. From a numerical simulation we find that the sublinear increase holds also in the case of preserved time-reversal symmetry. The quantity K deserves interest not only because it determines the quantum limit of the linewidth of laser radiation, via eq. (4). It is of fundamental significance in the general theory of scattering resonances, where it enters the width-to-height relation of resonance peaks and determines the scattering strength of a quasi-bound state with given decay rate Γ, via eq. (3). Because K is a measure for the non-orthogonality of the eigenmodes, this provides a way to obtain additional information about an externally probed open system.

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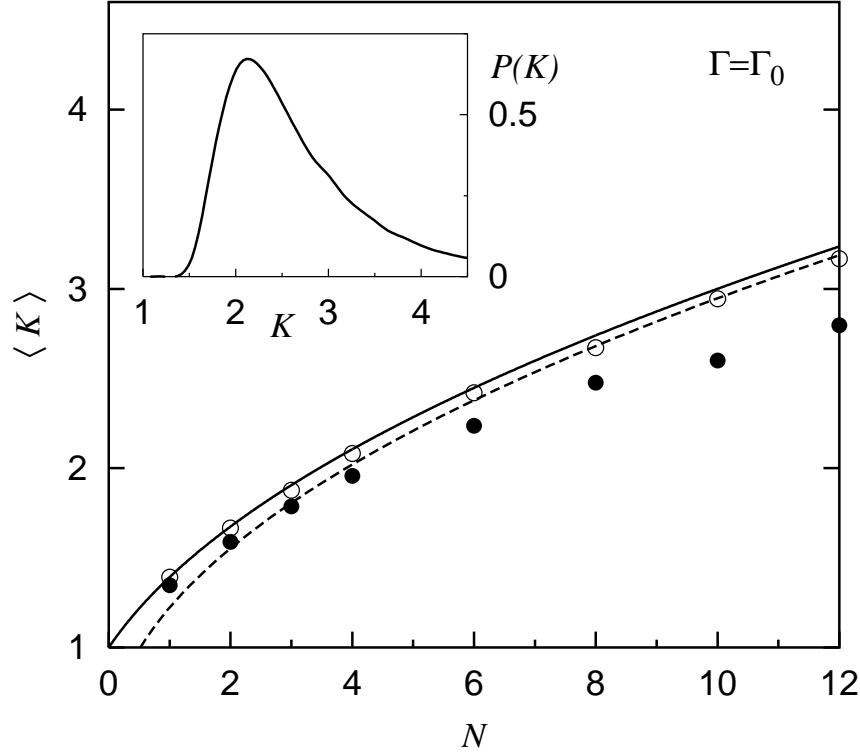[2] C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).

Fig. 2. – Average Petermann factor at $\Gamma = \Gamma_0$ as a function of N. The exact result (14) for broken time-reversal symmetry (solid curve) is compared with the result of a numerical simulation (open circles). Also shown is the simplified expression (19) for large N (dashed curve). The solid circles are the numerical result for preserved time-reversal symmetry. The inset shows the distribution of K at $\Gamma = \Gamma_0$ for $N = 10$ and broken time-reversal symmetry, following from the numerical simulation.

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